QPSK, OQPSK, CPM
Probability Of Error for AWGN and Flat Fading Channels [4]

16:332:546 Wireless Communication Technologies Spring 2005*
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Abstract

This article discusses QPSK, OQPSK, $\pi/4$ DQPSK and the trade offs involved. Later we discuss various CPM schemes and their relative bandwidth efficiencies. Probability of Bit error is discussed for various schemes assuming an AWGN channel. A model is devised, ignoring any changes in phase, to determine the error probabilities for flat fading channels. The article ends with a brief discussion of non-coherent detection.

1 QPSK and Offset QPSK Signaling

QPSK or Quadrature Phase Shift Keying, involves the splitting of a data stream $m_k(t) = m_0, m_1, m_2, \ldots$, into an in-phase stream $m_I(t) = m_0, m_2, m_4, \ldots$ and a quadrature stream $m_Q(t) = m_1, m_3, m_5, \ldots$. Both the streams have half the bit rate of the data stream $m_k(t)$, and modulate the cosine and sine functions of a carrier wave simultaneously. As a result, phase changes across intervals of $2T_b$, where T_b is the time interval of a single bit (the $m_k(t)$s). The phase transitions can be as large as $\pm\pi$ as shown in Figure 1.

Sudden phase reversals of $\pm\pi$ can throw the amplifiers into saturation. As shown in Figure 2 [1], the phase reversals of $\pm\pi$ cause the envelope to go to zero momentarily. This may make us susceptible to *non-linearities* in amplifier circuitry. The above may be prevented using linear amplifiers but they are more *expensive and power consuming*. A solution to the above mentioned problem is the use of OQPSK.

*Taught by Dr. Narayan Mandayam, Rutgers University.

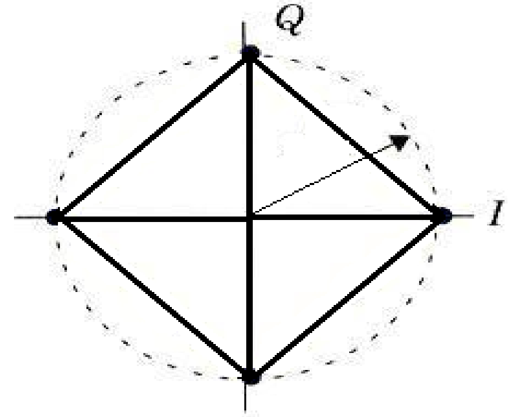

Figure 1: The figure shows a QPSK constellation. The dark black lines show all possible phase changes.

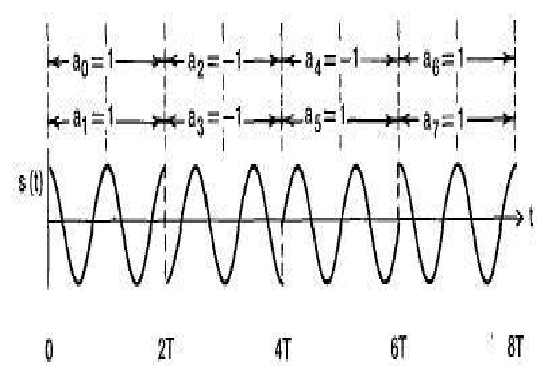

Figure 2: The figure shows a QPSK waveform. As is seen across the dotted line corresponding to a phase shift of π, the envelope reduces to zero temporarily.

OQPSK modulation is such that phase transitions about the origin are avoided. The scheme is used in IS-95 handsets. In OQPSK the pulse streams $m_I(t) = m_0, m_2, m_4, \ldots$ and $m_Q(t) = m_1, m_3, m_5, \ldots$ are *offset in alignment*, in other words are staggered, by one bit period (half a symbol period). Figure 3 [2], shows the staggering of the data streams in time. Figure 4 [1], shows the OQPSK waveform undergoing a phase shift of $\pm\pi/2$. The result of *limiting* the phase shifts to $\pm\pi/2$ is that the envelope will not go to zero as it does with QPSK.

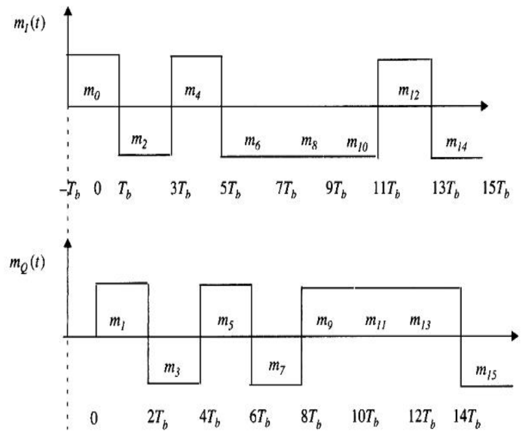

Figure 3: The figure shows the staggering of the in phase and quadrature modulated data streams in OQPSK. The staggering restricts the phase changes to ±90 as shown in Figure 4.

In OQPSK, the phase transitions take place every T_b seconds. In QPSK the transitions take place every $2T_b$ seconds.

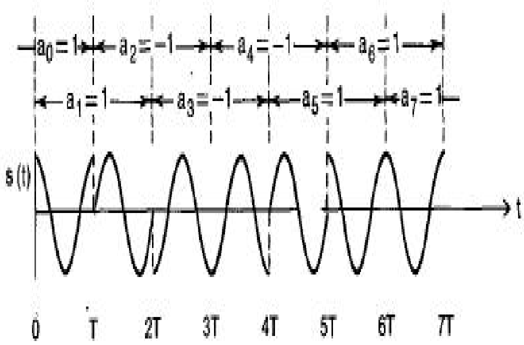

Figure 4: The figure shows a QPSK waveform. As is seen across the dotted lines the phase changes are of $\pm\pi/2$.

The OQPSK constellation is as shown in Figure 5.

2 $\pi/4$ DQPSK Signaling

The signaling is a compromise between QPSK and OQPSK in that the maximum transitions are allowed to be $\pm 3\pi/4$. The scheme is used in North American TDMA (IS-136). Figure 6

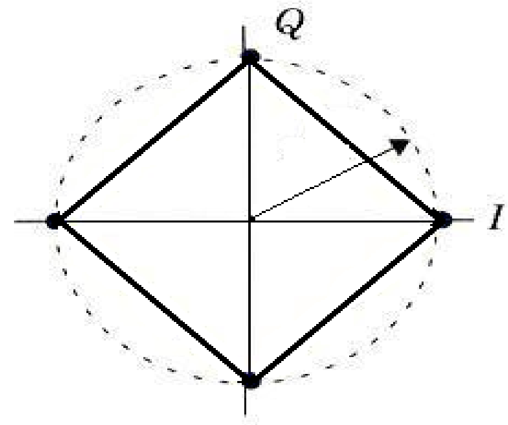

Figure 5: The figure shows a OQPSK constellation. The dark black lines show all possible phase changes. The signal space is the same as in the case of QPSK, though phase changes are restricted to ±90.

shows an example set of phase values that we may choose.

Information bits m_{Ik}, m_{Qk}	Phase shift ϕ_k
1 1	$\pi/4$
0 1	$3\pi/4$
0 0	$-3\pi/4$
1 0	$-\pi/4$

Figure 6: The figure shows an allowable table of phase transitions in $\pi/4$ DQPSK. The maximum phase translation allowed is ±135.

Figure 7 [2] gives two possible constellations and their all possible phase transitions.

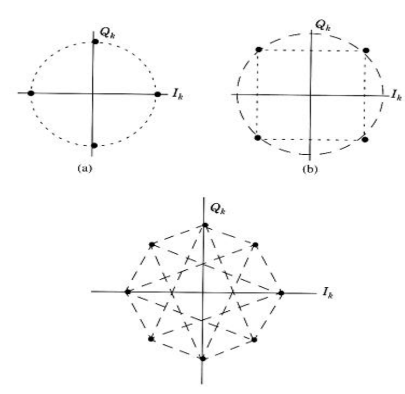

Figure 7: $\pi/4$ DQPSK constellations and all possible phase phase translations.

3 CPM

Constant envelope and very good spectral characteristics make CPM, Continuous Phase Modulation, a preferred choice in wireless communications. The complex baseband equivalent is given by

$$v(t) = A\ exp\Big\{j2\pi k_f \int_{-\infty}^{t} \sum_n x_n h_f(\tau - nT)\ d\tau\Big\} \\ = A\ exp\{j\phi(t)\} \tag{1}$$

where A is the amplitude, k_f is the peak frequency deviation, $h_f(t)$ is the frequency shaping pulse and T is the symbol duration. The symbol source sequence is $\{x_n\} = \{\pm1, \pm3, \pm5, \ldots, \pm(M-1)\}$, where M is the alphabet size.

$$\begin{aligned} \phi(t) =\ & 2\pi k_f \int_{-\infty}^{kT} \sum_{n=-\infty}^{k-1} x_n h_f(\tau - nT)\ d\tau \\ & + 2\pi k_f \int_{kT}^{t} h_f(\tau - kT)\ d\tau \\ & \text{where, } kT \le t \le (k+1)T \end{aligned} \tag{2}$$

If $h_f(t) = 0$ for $t > T$, the CPM signal is called *full response* CPM. If $h_f(t) \neq 0$ for $t > T$, the CPM signal is called *partial response* CPM. Using the *standard form* of representation of a baseband signal,

$$v(t) = A \sum_k b(t - kT, \underline{x}_k) \tag{3}$$

$$b(t, \underline{x}_k) = u_T(t) \exp\Big\{j\Big[\beta(\tau) \sum_{-\infty}^{k-1} x_n + x_k \beta(t)\Big]\Big\} \tag{4}$$

In the above equation, $\beta(\tau)\sum_{-\infty}^{k-1} x_n$ is the accumulated excess phase (memory) and $x_k\beta(t)$ is the excess phase for the current symbol.

$$\beta(t) = \begin{cases} 0 & t < 0 \\ 2\pi k_f \int_0^t h_f(\tau)\ d\tau & 0 \le t \le T \\ \beta(T) & t \ge T \end{cases} \tag{5}$$

Two terms that characterize CPM are the *Average Frequency Deviation* $\bar{k}_f$ and the *Modulation Index* h.

$$\bar{k}_f = (k_f/T) \int_0^T h_f(\tau)\ d\tau \tag{6}$$

$$h = \beta(\tau)/\pi = 2\bar{k}_f T \tag{7}$$

3.1 CPFSK

A conventional FSK signal is generated by shifting the carrier by an amount $f_n = 1/2\ \Delta f I_n, I_n = \pm1, \pm3, \ldots, \pm(M-1)$, to reflect the digital information that is being transmitted. The type of FSK signal is memoryless. Further, the switching from one frequency to another may be accomplished by having $M = 2^k$ separate oscillators tuned to the desired frequencies and selecting one of the M frequencies according to the k-bit symbol that is to be transmitted in a signal interval of duration $T = k/R$ seconds. However, such abrupt switching from one oscillator output to another in successive signaling intervals results in relatively large spectral side lobes outside of the main spectral band of the signal and, consequently, this method requires a large frequency band for transmission of the signal. To avoid the use of signals having large spectral side lobes, the information bearing signal frequency modulates a single carrier whose frequency is changed continuously. The resulting frequency modulated signal is phase continuous and, hence, it is called *continuous-phase* FSK.

$$h_f(t) = u_T(t) \tag{8}$$

$$\bar{k}_f = k_f, h = 2k_f T \tag{9}$$

$$\beta(t) = \begin{cases} 0 & t < 0 \\ 2\pi k_f t = \pi h t/T & 0 \le t \le T \\ \pi h & t \ge T \end{cases} \tag{10}$$

CPM signals are usually described by sketching the excess phase $\phi(t)$.

$$\phi(t) = \beta(T) \sum_{n=-\infty}^{k-1} x_n + \; x_k \beta(t - kT) \text{ for all } \{x_n\} \tag{11}$$

$\phi(t)$ is plotted for Binary CPFSK, $\{x_n\} = \{-1, 1\}$ in Figure 8 [3].

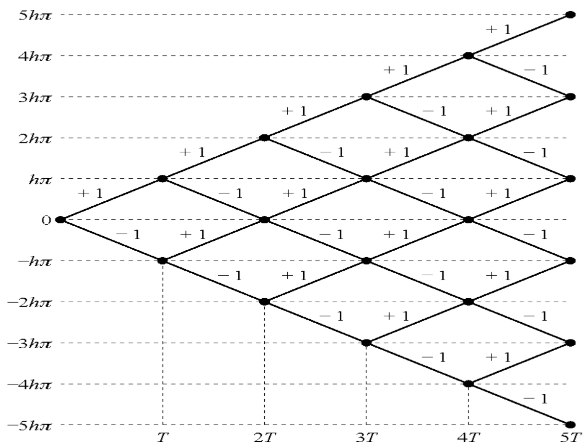

Figure 8: Phase trajectory for binary CPFSK.

The figure corresponds to a rectangular pulse representing a bit. Hence, the phase changes at a constant rate (straight line). Figure 9 compares the phase changes between a raised cosine pulse and a rectangular pulse shape.

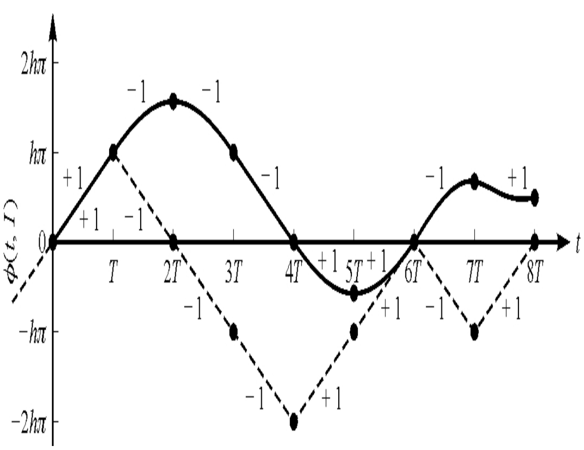

Figure 9: Phase trajectory for binary CPFSK using a rectangular pulse (dotted) and a raised cosine pulse. The I is the $\{x_n\}$

3.1.1 MSK

MSK, Minimum shift keying is a special case of binary CPFSK with $h = 0.5$.

$$\beta(t) = \begin{cases} 0 & t < 0 \\ 2\pi k_f t = \pi t/2T & 0 \leq t \leq T \\ 0.5\pi & t \geq T \end{cases} \tag{12}$$

Therefore, the carrier phase during the interval $kT \leq t \leq (k+1)T$ is given by,

$$\begin{aligned} \phi(t) &= 2\pi f_c t + \pi/2 \sum_{n=-\infty}^{k-1} x_n + 0.5\pi \; x_k \; ((t - kT)/T) \\ &= \Big(2\pi f_c + \pi x_k/2T\Big)t + \pi/2 \sum_{n=-\infty}^{k} x_n - \pi/2 \; x_k \end{aligned} \tag{13}$$

The MSK bandpass waveform is then given as

$$\begin{aligned} s(t) &= A\cos\Big[(2\pi f_c + \pi x_k/2T)t + \\ &\quad \pi/2 \sum_{n=-\infty}^{k-1} x_n \; - \; \pi k/2 \; x_k\Big] \\ &\quad \text{where } kT \leq t \leq (k+1)T \\ &= A\cos\Big[2\pi(f_c + x_k/4T)t + \\ &\quad \pi/2 \sum_{n=-\infty}^{k-1} x_n \; - \; \pi k/2 \; x_k\Big] \\ &\quad \text{where } x_k \epsilon (-1, 1) \end{aligned} \tag{14}$$

Since, x_k can be ± 1, two different frequencies are modulated for ± 1. The frequencies are $f_c \pm 1/4T$. Therefore, the difference between the frequencies is $1/2T$ which is the minimum frequency separation required to ensure orthogonality between two sinusoids of duration T, assuming coherent demodulation. The above is the reason why the scheme is called *minimum* shift keying.

The Power Spectral Density of MSK is shown in the Figure 10 for different pulse shapes. Figure 11 compares the spectra of MSK and OQPSK. Note that the main lobe of MSK is 50% wider than that for OQPSK. However, the side lobes in MSK fall off considerably faster, making MSK more bandwidth efficient. Even greater efficiency than MSK can be achieved by further reducing h. However, the FSK signals will no longer be orthogonal and there will be an increase in the error probability.

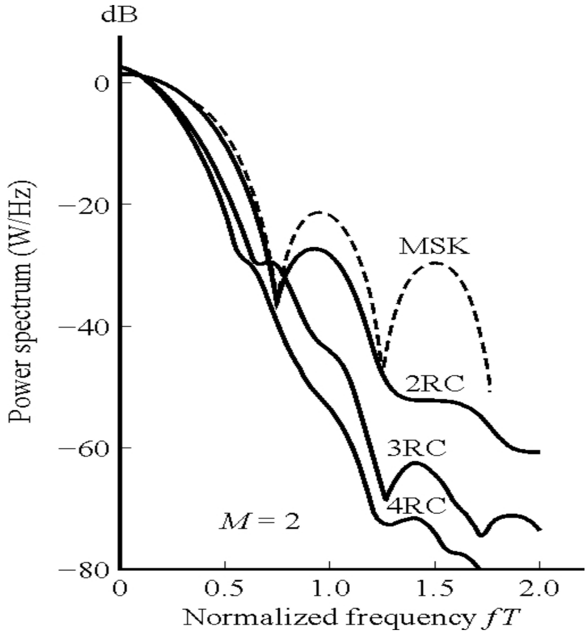

Figure 10: PSD for MSK. The dotted curve corresponds to MSK. The other curves correspond to a raised cosine pulse, partial response CPFSK with $h = 0.5$, lasting for $2T$,$3T$,$4T$. It is clearly seen that as the spreading in time is increased, the bandwidth efficiency increases.

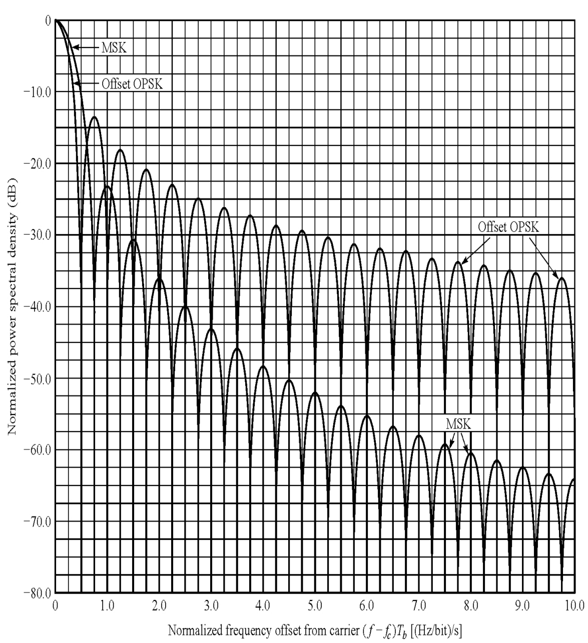

Figure 11: PSD comparison of MSK and OQPSK.

3.2 Partial Response CPM

The idea is to make $h_f(t)$ of duration greater than T.

$$h_f(t) = h_f(t)\ u_{kT}(t)$$
$$\text{where } u_{kT} = \sum_{k=0}^{K-1} u_T(t - kT) \tag{15}$$

3.2.1 GMSK

Pass the rectangular pulse $h_f(t)$ through a premodulation filter given as

$$H(f) = exp\Big\{ -(f/B)^2 \ln 2/2 \Big\} \tag{16}$$

B is the bandwidth of the filter. H(f) is bell shaped about $f = 0$. Therefore, the name Gaussian MSK. BT is used to parameterize GMSK schemes. B is the bandwidth of the premodulation filter defined above and T is the symbol duration. The next two figures make it amply clear in both the time and frequency domain, that decreasing BT improves spectral occupancy.

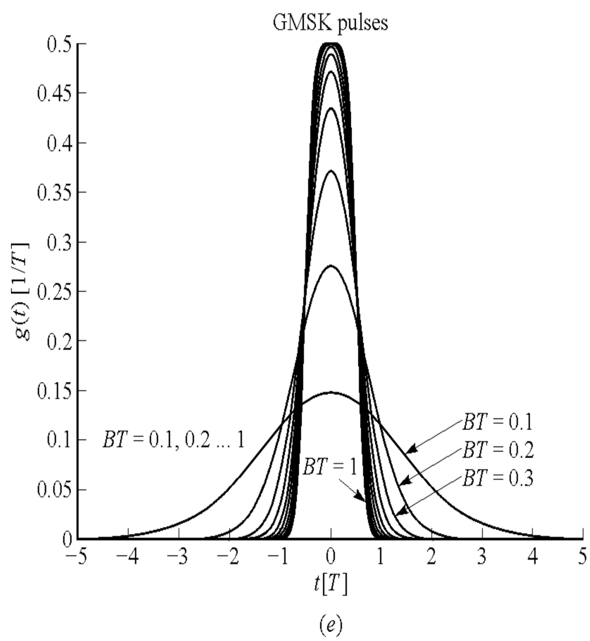

Figure 12: GMSK pulses for different BT.

From Figure 12, we observe that when $BT = 0.3$, the GMSK pulse may be truncated at $|t| = 1.5T$ with a relatively small error incurred. GMSK with $BT = 0.3$ is used in GSM.

The decrease in spectral occupancy is accompanied by increase in ISI, as we no longer adhere to the Nyquist Criterion. To counter the ISI, GMSK requires equalization. Thus, we can say that GMSK is a bandwidth efficient scheme but not a power efficient one. The table below shows occupied RF bandwidth for GMSK and MSK as a fraction of R_b, the bit rate, containing a given percentage of power. Notice that GMSK is spectrally tighter than MSK.

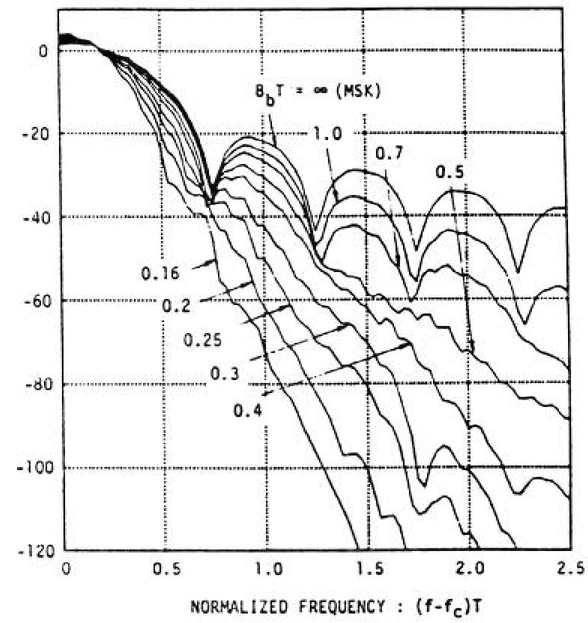

Figure 13: Power spectral density for a GMSK signal.

BT	90%	99%	99.9%	99.99%
0.2 GMSK	0.52	0.79	0.99	1.22
0.25 GMSK	0.57	0.86	1.09	1.37
0.5 GMSK	0.69	1.04	1.33	2.08
MSK	0.78	1.20	2.76	2

4 M-PSK Bandwidth/Power

The MPSK waveform is given by,

$s_i(t) = \ (2E_s/T_s)^{0.5} \cos(2\pi f_c(t) + 2\pi/M \ (i-1))$
where $0 \leq t \leq T_s$ and $T_s = (\log_2 M)T_b$
$E_s = E_b \log_2 M \ and T_b$ is the energy per symbol. (17)

The table below highlights the bandwidth and power efficiency of M-PSK signals. Power efficiency η_p is defined as E_b/N_o required for $P_e = 10^{-6}$. The bandwidth is the first null bandwidth.

M	2	4	8	16	32	64
η_b	0.5	1	1.5	2	2.5	3
E_b/N_o	10.5	10.5	14	18.5	23.4	28.5

5 Optimum Receivers AWGN

For a signal transmitted over an AWGN channel [3], either a correlation demodulator or a matched filter demodulator produces the vector $\underline{r} = [r_1 r_2 \ldots r_N]$, which contains all the relevant information in the received signal waveform. Once the vector $\underline{r}$ has been received, an optimum decision needs to be made regarding which signal $\underline{s}_m$ was transmitted, given that $\underline{r}$ has been received where $1 \leq m \leq M$. The decision criterion is based on selecting the signal corresponding to the maximum of the set of a posteriori probabilities $\{P(\underline{s}_m|\underline{r})\}$. The decision criterion is called the MAP (maximum a posteriori probability) criterion. It can be proved that the criterion minimizes the probability of error and therefore a detector that implements it is known as the optimum detector.

Using Bayes' rule, the a posteriori probabilities can be expressed as $P(\underline{s}_m|\underline{r}) = p(\underline{r}|\underline{s}_m)P(\underline{s}_m)/p(\underline{r})$, where $p(\underline{r}|\underline{s}_m)$ is the conditional PDF of the observed vector given that $\underline{s}_m$ was transmitted and $P(\underline{s}_m)$ is the a priori probability of the mth signal being transmitted.

If we further assume that all M signals are equally probable a priori, the optimum detection rule reduces to finding the transmitted signal that maximizes $p(\underline{r}|s_m)$. The criterion is also known as the ML (maximum likelihood) criterion.

For an AWGN channel,

$$p(\underline{r}|\underline{s}_m) = (\pi N_0)^{-N/2} exp\Big[(-1/N_0) \ \sum_{k=1}^{N} (r_k - s_{mk})^2\Big] \qquad 0 \leq m \leq M \tag{18}$$

Therefore,

$$\ln p(\underline{r}|\underline{s}_m) = (-N/2) \ \ln(\pi N_0) - 1/N_0 \ \sum_{k=1}^{N} (r_k - s_{mk})^2 \tag{19}$$

The maximum of $\ln p(\underline{r}|\underline{s}_m)$ over s_m is equivalent to finding the signal $\underline{s}_m$ that minimizes $\sum_{k=1}^{N}(r_k - s_{mk})^2$, which is the same as the Euclidean distance between the received vector and the vector $\underline{s}_m$.

Further, if we assume that all the $\underline{s}_m$ have the same energy, the criteria becomes selecting the vector $\underline{s}_m$ that has the maximum correlation with the received vector $\underline{r}$. Remember that the vector was received at the output of the matched filter/ correlation demodulator.

All the observations made above are assuming memoryless modulation. However, for signals that have memory, the maximum likelihood sequence detection algorithm must be used. The algorithm searches for the minimum Euclidean distance path through the trellis that characterizes the memory in the transmitted signal.

5.1 Probability Of Error

5.1.1 Binary FSK

It is given by $Q(\sqrt{E_b/N_o})$.

5.1.2 MSK

MSK has one of two possible frequencies over any symbol interval.

$$s(t) = A \cos\Big[(2\pi f_c + \pi x_k/2T)t + \pi/2 \sum_{n=-\infty}^{k-1} x_n - \pi/2\ kx_k\Big] \quad (20)$$

where $kT \le t \le (k+1)T$

Consider interval $0 \le t \le T_b$ and denote $A = \sqrt{2E_b/T_b}$. Then

$$\beta(t) = \begin{cases} \sqrt{2E_b/T_b}\cos(2\pi f_1 t + \theta(o)) & \text{symbol 1} \\ \sqrt{2E_b/T_b}\cos(2\pi f_2 t + \theta(o)) & \text{symbol 0} \end{cases} \quad (21)$$

On first guess one may conclude that MSK has the same error probability as BFSK, but since it has memory of phase it does better. Using phase trellis it can be shown that $P_e \approx Q(\sqrt{2E_b/T_b})$, the approximation is valid for high SNR values. Therefore, MSK is approximately same in BER performance as BPSK.

5.1.3 GMSK

The probability of error of GMSK can be shown to be $P_e \approx Q(\sqrt{2\alpha E_b/T_b})$ where α is a constant for a given BT_b. For example, for $BT_b = 0.25$, $\alpha \approx 0.68$. Similarly, for $BT_b = \infty$, $\alpha \approx 0.85$ which is the case of MSK.

We had earlier noted that GMSK improved bandwidth efficiency over MSK since it had a narrower main lobe and a faster roll-off of side bands. But from above it is clear that $P_e(GMSK) > P_e(MSK)$

6 B/W vs Power efficiency trade off

As BT_b decreases, bandwidth efficiency increases, but power efficiency decreases (because P_e increases).

7 Error Probabilities for Flat Fading Channels

Consider the transmitted waveform,

$$s_i(t) = \sqrt{2E_s/T_s}\cos(2\pi f_c t + 2\pi/M\ (i-1)), \quad 0 \le t \le T_s \quad (22)$$

7.1 Flat fading received model

$x(t) = g(t)s_i(t) + w(t)$, where $w(t)$ is AWGN. $g(t)$ is the attenuation in the amplitude of the signal due to fading. Assume that the channel is *flat and slow*. Therefore $T_s >> \sigma_\tau$ and $T_s << T_c$. Therefore $g(t)$ is effectively a constant over the symbol duration. Let $g(t) = \alpha$, then $x(t) = \alpha s_i(t) + w(t)$ for $0 \le t \le T_s$.

For a constant α the maximum likelihood decoding rule for optimum detection (assuming inputs are equiprobable) holds true. Therefore, the receiver structure remains the same. In general it can be shown that,

$$P_e \le \sum_{\substack{k=1 \\ k\ne i}} Q(\alpha d_{ik}/\sqrt{2N_0}) \quad (23)$$

Typically, α is Rayleigh or Rician distributed for non LOS and LOS situations respectively. Therefore the probability of error may be written as,

$$P_e \le \sum_{\substack{k=1 \\ k\ne i}} \int_0^\infty Q(\alpha d_{ik}/\sqrt{2N_0}) f_\alpha(\alpha)\ d\alpha \quad (24)$$

Consider $M = 2$, the SNR is given as

$\gamma_b = \alpha^2 E_b/N_0$. Let $\beta = \alpha^2$. If α is Rayleigh, β is exponential.

$$\bar{P}_e = \int_0^\infty Q(\sqrt{2\beta E_b/N_0}) f_\beta(\beta)\ d\beta \tag{25}$$

Rewrite, $\gamma_b = \beta E_b/N_0$. Then

$$E[\gamma_b] = \bar{\gamma}_b = E_b/N_0\ E[\beta] \tag{26}$$

$$f(\gamma_b) = (\bar{\gamma}_b)^{-1}\ exp(-\gamma_b/\bar{\gamma}_b)\ ,\ \gamma_b \geq 0 \tag{27}$$

Therefore,

$$\bar{P}_e = \int_0^\infty Q(\sqrt{2\gamma_b})(\gamma_b)^{-1} exp(-\gamma_b/\bar{\gamma}_b)\ d\gamma_b \tag{28}$$

Integrating by parts we get,

$$\bar{P}_e = 0.5 - (1/2\sqrt{\pi}) \int_0^\infty exp(-\gamma_b(1+\bar{\gamma}_b)/\bar{\gamma}_b)\ \gamma_b^{-0.5} d\gamma_b \tag{29}$$

Substituting,

$$z = \gamma_b(1+\bar{\gamma}_b)/\bar{\gamma}_b) \tag{30}$$

$$\bar{P}_e = 1/2 - 1/2\ (\pi(1+\bar{\gamma}_b)/\bar{\gamma}_b)^{-1} \int_0^\infty e^{-z}\ z^{-0.5}\ dz \tag{31}$$

$$\bar{P}_e = 1/2 - 1/2\ (\pi(1+\bar{\gamma}_b)/\bar{\gamma}_b)^{-1}\Gamma(1/2) \tag{32}$$

$$\bar{P}_e = 1/2 - 1/2\ ((1+\bar{\gamma}_b)/\bar{\gamma}_b)^{-1} \tag{33}$$

For high SNR ($\bar{\gamma}_b$) we can say that $P_e \propto (SNR)^{-1}$, unlike AWGN where they were exponentially related. Consider the probability of error, if using Binary FSK on a flat fading channel as modeled above.

$$P_e = 1/2 - 1/2\ ((2+\bar{\gamma}_b)/\bar{\gamma}_b)^{-1} \tag{34}$$

Therefore, coherent PSK is 3dB better than coherent FSK.

It is important to note that in the above model, we assumed that coherent detection was possible. That is why the phase was completely ignored in the model. For coherent detection to be possible in a fading channel, we need pilot signals.

7.2 Detection of signals with unknown phase

If we assume that the phase of the signal is not known at the receiver we will have to use *Non-Coherent* detection. Let the transmitted signal be,

$$s_i(t) = \sqrt{2E/T}\cos(2\pi f_i t),\ 0 \leq t \leq T \tag{35}$$

The received signal may be written as,

$$x(t) = \sqrt{2E/T}\cos(2\pi f_i t + \theta) + w(t) \tag{36}$$

where $w(t)$ is the AWGN and θ is the unknown phase. So we can assume that θ is a random variable uniformly distributed in the interval $[0, 2\pi]$. Rewriting, $x(t)$ as

$$x(t) = \sqrt{2E/T}\Big\{\cos(2\pi f_i t)\cos(\theta) - \sin(2\pi f_i t)\sin(\theta)\Big\} + w(t) \tag{37}$$

The signal may be received using a *Quadrature Receiver.*

7.3 Non Coherent Orthogonal Modulation

Assume that bit 1 is transmitted as $s_1(t)$ and bit 0 is transmitted as $s_2(t)$. As the modulation is orthogonal, $s_1(t)$ and $s_2(t)$ are orthogonal. We further assume that the received signal is $x(t)$.

$$x(t) = \begin{cases} g_1(t) + w(t) & \text{,for } 0 \leq t \leq T \text{, if 1 is Tx} \\ g_2(t) + w(t) & \text{,for } 0 \leq t \leq T \text{, if 0 is Tx} \end{cases} \tag{38}$$

We can further assume that $g_1(t)$ and $g_2(t)$ are orthogonal. The receiver is shown in Figure 14.

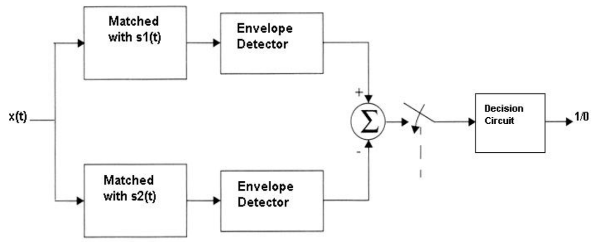

Figure 14: Non-Coherent detection. Each ARM of the receiver is a Quadrature Receiver.

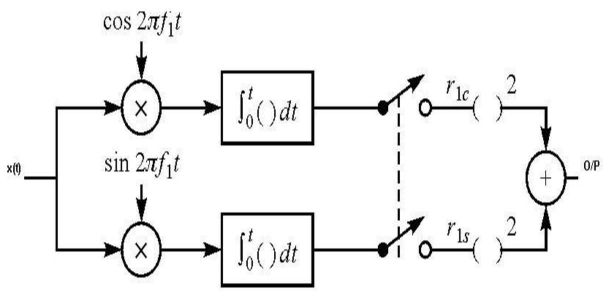

Figure 15: A Quadrature Receiver.

References

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